

On the geometry and topology of manifolds of positive bi-Ricci curvature

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1 Introduction

In [ShY1], we introduced the concept of bi-Ricci curvature and initiated the study of manifolds of positive bi-Ricci curvature. We obtained a size estimate for stable minimal hypersurfaces in 3, 4 and 5 dimensional manifolds of positive bi-Ricci curvature. As a consequence, a homology radius estimate for the manifolds was derived.

In the present paper, we obtain a number of results regarding bi-Ricci curvature. First, we extend the size estimate in [ShY1] to six or less dimensional manifolds of positive harmonic bi-Ricci curvature. Here, “harmonic bi-Ricci curvature” is a variant of bi-Ricci curvature. Indeed, it is one of the “weighted bi-Ricci curvatures”.

The above extension is achieved by choosing a certain conformal factor in our computations to be more general than in [ShY1]. On the other hand, we adopt a strategy for deriving our size estimates which somewhat differs from that in [ShY1]. Indeed, we first introduce the concept of “conformal Ricci curvature”, and present a generalized Bonnet-Myers theorem about manifolds of positive conformal Ricci curvature. The desired size estimates for stable minimal hypersurfaces then follow as a consequence of this theorem (or its underlying estimate). This way, a general geometric background for the topic of bi-Ricci curvature can be seen. We expect further applications of the generalized Bonnet-Myers theorem. (See [ShY2] for some closely related results on Riemannian and Lorentzian warped products and their applications

to general relativity.) On the other hand, we view conformal Ricci curvature as an interesting independent subject.

As the second topic of this paper, we derive more geometric and topological consequences of positive (harmonic) bi-Ricci curvatures. On the other hand, we show that connected sums of manifolds of positive bi-Ricci curvatures admit metrics of positive bi-Ricci curvatures. (Note that similar connected sum theorems hold for scalar curvature [ScY], [GL] and isotropy curvature [H], [MW].) It remains to be investigated how a general topological classification scheme for manifolds of positive bi-Ricci curvature should proceed. But our topological results already provide substantial information.

Note that besides its own interest, the subject of positive bi-Ricci curvature has the potential use of providing new information on positive sectional curvature. This is suggested e.g. by Observation 2 in the sequel. We hope to be able to report on this in the near future.

The last topic of this paper is extension of our techniques to minimal surfaces of higher codimensions. Diameter estimates are derived for 2-dimensional stable minimal surfaces in spaces of dimensions up to 9, and 3-dimensional stable minimal surfaces in spaces of dimension 5, under suitable positive curvature assumptions and an "almost flatness" condition on the normal bundle. Currently, we are trying to find ways to handle this almost flatness condition. We believe that further understanding of positive curvatures can be achieved this way.

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2 Conformal Ricci curvature

We start with the following definition.

Definition 1 Let (N, h) be a Riemannian manifold of dimension n with metric h . For a positive smooth function f on N and a positive constant σ we define the *conformal Ricci tensor* of N associated with the conformal factor f and the weight σ to be

$$Ric^{(f,\sigma)} \equiv Ric - \sigma(f^{-1}\Delta f)h. \quad (2.1)$$

The conformal Ricci curvature (associated with conformal factor f and weight σ) in the direction of a unit tangent vector v is then defined to be $Ric^{(f,\sigma)}(v, v)$. We say that (N, h) has positive conformal Ricci curvature if there is a conformal factor f and a weight σ such that the conformal Ricci tensor $Ric^{(f,\sigma)}$ is positive definite.

A more general version of this definition will be given in the last section. The terminology in this definition is motivated by the transformation laws for Ricci curvature and scalar curvature under conformal change of metric. Let η be a positive

smooth function on N and $\tilde{h} = \eta^2 h$ the corresponding conformal metric. The said transformation laws are (see e.g. [B])

$$\tilde{Ric} = Ric - (\Delta \ln \eta)h - (n-2)(\nabla d(\ln \eta) - d \ln \eta \otimes d \ln \eta) - (n-2)|\nabla \ln \eta|^2 h, \quad (2.2)$$

and

$$\tilde{R} = \eta^{-2}(R - 2(n-1)\Delta \ln \eta - (n-1)(n-2)|\nabla \ln \eta|^2), \quad (2.3)$$

where $\tilde{Ric} = Ric_{\tilde{h}}$, $\tilde{R} = R_{\tilde{h}}$ (R denotes the scalar curvature of h), and Δ is the trace Laplacian. If we set $\eta = f^{2/(n-2)}$, then (as is well-known)

$$\tilde{R} = f^{-4/(n-2)}(R - \frac{4(n-1)}{n-2}f^{-1}\Delta f).$$

This is precisely the trace of the conformal Ricci tensor $Ric^{(f, 4(n-1)/(n-2))}$ with respect to the conformal metric $f^{4/(n-2)}h$.

Using (2.2), we can obtain another expression for conformal Ricci curvature. For given f and σ , set $\tilde{h} = f^{2\sigma}h$. Let v be a unit tangent vector with respect to h . Consider the geodesic $\tilde{\gamma}$ with respect to \tilde{h} which has $\tilde{v} = f^{-1}v$ as the initial tangent vector. Let $\gamma = \gamma(s)$ be the reparametrization of $\tilde{\gamma}$ which has unit speed measured in h . It is easy to see that the geodesic equation for $\tilde{\gamma}$ can be rewritten as follows

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial s} = \sigma(\nabla \ln f)^\perp, \quad (2.4)$$

where the \perp denotes the projection to the orthogonal complement of $\frac{\partial \gamma}{\partial s}$. Using this equation we derive

$$\begin{aligned} \nabla d(\ln f)\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial s}\right) &= \frac{\partial^2 \ln f}{\partial s^2} - \left(\nabla_{\frac{\partial}{\partial s}} \frac{\partial \gamma}{\partial s}\right) \ln f \\ &= \frac{\partial^2 \ln f}{\partial s^2} - |(\nabla \ln f)^\perp|^2. \end{aligned}$$

Combining this with (2.2) (setting $\eta = f^\sigma$ there) then yields

$$\tilde{Ric}(v, v) = Ric(v, v) - \Delta \ln f^\sigma - (n-2)\frac{\partial^2 \ln f^\sigma}{\partial s^2}, \quad (2.5)$$

whence

$$Ric^{(f, \sigma)}(v, v) = \tilde{Ric}(v, v) + (n-2)\sigma \frac{\partial^2 \ln f}{\partial s^2} - \sigma |\nabla \ln f|^2. \quad (2.6)$$

Theorem 1 (*generalized Bonnet-Myers theorem*) *Let (N, h) be connected and complete. Assume that there are a conformal factor f and weight σ , along with positive constants κ, ε , such that*

$$Ric^{(f, \sigma)}(v, v) + \sigma |\nabla \ln f|^2 - \left(\frac{(n-1)}{4} + a(n)\varepsilon \right) \sigma^2 |v \ln f|^2 \geq \kappa \quad (2.7)$$

for all unit tangent vectors v , where $a(3) = 0, a(n) = 1$ for $n \neq 3$. Then N is compact. Indeed, we have

$$diam(N, h) \leq \sqrt{n-1 + \frac{(n-3)^2}{4\varepsilon} \frac{\pi}{\sqrt{\kappa}}}. \quad (2.8)$$

Consequently, the universal cover of N is also compact, and hence N has finite fundamental group.

Corollary 1 *Assume that for a factor f and a weight $\sigma < \frac{4}{n-1}$ ($\sigma \leq \frac{4}{n-1}$ if $n = 3$), along with a positive constant κ , there holds*

$$Ric^{(f, \sigma)}(v, v) \geq \kappa. \quad (2.9)$$

Then N is compact and has finite fundamental group. Indeed, we have

$$diam(N, h) \leq \sqrt{n-1 + \frac{(n-3)^2}{\frac{4}{\sigma} - n + 1} \frac{\pi}{\sqrt{\kappa}}}. \quad (2.10)$$

if $\sigma < \frac{4}{n-1}$, and

$$diam(N, h) \leq \sqrt{n-1} \frac{\pi}{\sqrt{\kappa}} \quad (2.11)$$

if $\sigma \leq \frac{4}{n-1}$ and $n = 3$.

From the viewpoint of applications, the following more general version of Theorem 1 is noteworthy.

Theorem 1' *If we replace the constant κ in the condition (2.7) of Theorem 1 by the following quantity*

$$\frac{\kappa}{1 + dist(\cdot, p_0)^\delta} \quad (2.12)$$

for positive constants κ, δ with $\delta < 2$ and a point $p_0 \in N$, then we can still conclude that N is compact and has finite fundamental group. The diameter of (N, h) can be estimated as follows

$$diam(N, h) \leq 2\theta^{-1} \left((n-1 + \frac{(n-3)^2}{4\varepsilon}) \frac{\pi^2}{\kappa} \right),$$

where θ is the increasing function

$$\theta(t) = \frac{t^2}{1 + t^\delta}.$$

Similarly, we have

Corollary 1' *If we replace the curvature condition in Corollary 1 by the following*

$$Ric^{(f,\sigma)}(v, v) \geq \frac{\kappa}{1 + dist(\cdot, p_0)^\delta} \quad (2.13)$$

for positive constants κ, δ with $\delta < 2$ and a point $p_0 \in N$, then we can still conclude that N is compact and has finite fundamental group. There holds

$$diam(N, h) \leq 2\theta^{-1}\left((n-1) + \frac{(n-3)^2}{\frac{4}{\sigma} - n + 1} \frac{\pi^2}{\kappa}\right) \quad (2.14)$$

if $\sigma < \frac{4}{n-1}$, and

$$diam(N, h) \leq 2\theta^{-1}\left((n-1) \frac{\pi^2}{\kappa}\right). \quad (2.15)$$

if $\sigma \leq \frac{4}{n-1}$ and $n = 3$.

Remark 1 The classic Bonnet-Myers theorem follows from Theorem 1 by taking $f \equiv 1, \sigma = 1$ and ε arbitrarily large.

For convenience, we formulate two lemmas from which Theorem 1 will follow. Consider a general manifold (N, h) , a conformal factor f , a weight σ and the associated conformal metric $\tilde{h} = f^{2\sigma}h$. Let $\tilde{\gamma}$ be a unit speed geodesic in N with respect to \tilde{h} and $\gamma : [0, 1] \rightarrow N$ its reparametrization which has unit speed measured in h .

Lemma 1 *Assume that $\tilde{\gamma}$ minimizes length up to second order measured in \tilde{h} while its endpoints are kept fixed. Then we have*

$$\begin{aligned} (n-1) \int \left(\frac{d\phi}{ds}\right)^2 + \frac{n-1}{4} \int \phi^2 \left(\frac{d \ln f^\sigma}{ds}\right)^2 + (3-n) \int \phi \frac{d\phi}{ds} \frac{d \ln f^\sigma}{ds} \\ \geq \int \phi^2 Ric^{(f,\sigma)}(v, v) + \frac{1}{\sigma} \int \phi^2 |\nabla \ln f^\sigma|^2 \end{aligned}$$

for all smooth functions ϕ on $[0, l]$ with zero boundary values, where the element ds in the integrals is omitted.

Proof By the assumption on $\tilde{\gamma}$, there holds

$$(n-1) \int \left(\frac{d\phi}{d\tilde{s}}\right)^2 d\tilde{s} \geq \int \phi^2 Ric_{\tilde{h}}\left(\frac{\partial\tilde{\gamma}}{\partial\tilde{s}}\right) d\tilde{s}, \quad (2.16)$$

where ϕ is an arbitrary smooth function on $[0, \tilde{l}]$ with zero boundary values.

Setting $\eta = f^\sigma$ and applying (2.6) we infer

$$(n-1) \int \left(\frac{d\phi}{ds}\right)^2 \eta^{-1} \geq \int \eta^{-1} \phi^2 \left\{ Ric^{(f,\sigma)}\left(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial s}\right) - (n-2)\sigma \frac{\partial^2 \ln f}{\partial s^2} - \frac{1}{\sigma} |\nabla \ln \eta|^2 \right\}.$$

If we replace ϕ by $\phi\eta^{1/2}$, then this inequality becomes

$$\begin{aligned} (n-1) \int \left(\frac{d\phi}{ds}\right)^2 + \frac{n-1}{4} \int \phi^2 \left(\frac{d \ln \eta}{ds}\right)^2 + (n-1) \int \eta^{-1} \phi \frac{d\phi}{ds} \frac{d\eta}{ds} \\ \geq \int \phi^2 \left\{ Ric^{(f,\sigma)}\left(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial s}\right) - (n-2) \frac{\partial^2 \ln \eta}{\partial s^2} - \frac{1}{\sigma} |\nabla \ln \eta|^2 \right\}. \end{aligned}$$

Applying the following identity we then arrive at the desired inequality

$$\int_0^l \phi^2 \frac{d^2 \ln \eta}{ds^2} = -2 \int_0^l \phi \frac{d\phi}{ds} \frac{d \ln \eta}{ds}.$$

□

Lemma 2 *Let $\tilde{\gamma}$ satisfy the condition in Lemma 1 and $\gamma : [0, l] \rightarrow N$ be as before. If the curvature condition (2.7) holds, then the inequality (2.8) holds with $\text{diam}(N, h)$ replaced by l . Consequently, if the curvature condition (2.9) holds with $\sigma < \frac{4}{n-1}$ ($\sigma \leq \frac{4}{n-1}$ if $n = 3$), then the inequality (2.10) ((2.11), if $\sigma \leq \frac{4}{n-1}$ and $n = 3$) holds with $\text{diam}(N, h)$ replaced by l .*

Lemma 2' *Let $\tilde{\gamma}$ and γ be as above. If the curvature condition (2.7) holds with κ replaced by (2.12) (with $\delta < 2$), then we have*

$$l \leq \theta_{\gamma(0)}^{-1} \left((n-1 + \frac{(n-3)^2}{4\epsilon}) \frac{\pi^2}{\kappa} \right),$$

where

$$\theta_p(t) = \frac{t^2}{1 + (\text{dist}(p_0, p) + t)^\delta}.$$

Similarly, if the curvature condition (2.13) holds, then the inequality (2.14) (if $\sigma < \frac{4}{n-1}$) or (2.15) (if $\sigma \leq \frac{4}{n-1}$ and $n = 3$) holds with $\text{diam}(N, h)$ replaced by l , θ replaced by $\theta_{\gamma(0)}$, and the factor 2 removed.

With slight modifications, the following proof also applies to Lemma 2'. We leave the details to the reader.

Proof of Lemma 2 Applying Lemma 1 and the assumption on the conformal Ricci curvature we deduce

$$(n-1) \int \left(\frac{d\phi}{ds}\right)^2 + (3-n) \int \phi \frac{d\phi}{ds} \frac{d \ln f^\sigma}{ds} - a(n)\varepsilon \int \phi^2 \left(\frac{d \ln f^\sigma}{ds}\right)^2 \geq \kappa \int \phi^2.$$

If $n = 3$, this implies

$$(n-1) \int \left(\frac{d\phi}{ds}\right)^2 \geq \kappa \int \phi^2.$$

Choosing $\phi(s) = \sin \frac{\pi}{l}s$ we then deduce

$$l \leq \sqrt{\frac{n-1}{\kappa}} \pi.$$

If $n \neq 3$, we apply Yang's inequality and deduce

$$(n-1 + \frac{(n-3)^2}{4\varepsilon}) \int \left(\frac{d\phi}{ds}\right)^2 \geq \kappa \int \phi^2,$$

which leads to the desired estimate. □

Proof of Theorem 1 Let p be a point in N . Consider an arbitrary $r > 0$ such that the boundary $\partial B_r(p)$ of the geodesic ball $B_r(p)$ with respect to h is nonempty. We can find a minimizing geodesic $\tilde{\gamma}$ with respect to \tilde{h} , which runs from p to $\partial B_r(p)$. Let $\gamma : [0, l] \rightarrow N$ be its reparametrization which has unit speed measured in h . Since $r \leq l$, and p, r are arbitrary, Lemma 2 implies the desired diameter estimate. □

Proof of Theorem 1' We choose $p = p_0$ in the above proof (replacing Lemma 2 by Lemma 2'). For arbitrary $p, q \in N$ we then use the triangular inequality to estimate $\text{dist}(p, q)$. □

3 Minimal hypersurfaces

In this section, we apply Theorem 1 and Theorem 1' (or Lemma 2 and Lemma 2') to study stable minimal hypersurfaces in a Riemannian manifold. Let (M, g) be a

Riemannian manifold of dimension $m = n + 1$ with metric g . Consider a smoothly immersed minimal hypersurface $S \subset M$ and a bounded domain Ω in S such that $\bar{\Omega} \subset \overset{\circ}{S}$. The Jacobi operator L associated with the second variation of area for S is given by

$$L\phi = -\Delta_S \phi - |A|^2 \phi - Ric(\nu)\phi, \quad (3.1)$$

where ν denotes a unit normal of S , A the second fundamental form of S and ϕ a smooth function on S . Let $\lambda = \lambda_\Omega$ be the first eigenvalue of L on Ω for the Dirichlet boundary value problem, and f a corresponding nonnegative first eigenfunction, i.e. $Lf = \lambda f$. Then $f > 0$ in Ω . We consider the Riemannian manifold $N = \Omega$ with metric $h = g|_S$. Choosing the eigenfunction f as the conformal factor and an arbitrary weight σ , we proceed to compute conformal Ricci curvature $Ric_S^{(f,\sigma)}$, where we use the subscript S to distinguish from the Ricci tensor Ric of (M, g) .

Computing at a fixed point, we choose an orthonormal tangent base e_1, \dots, e_{n+1} such that $e_{n+1} = \nu$ and A is diagonalized in the base e_1, \dots, e_n . For an arbitrary unit tangent vector v of S at this point, we write

$$v = \sum_{1 \leq i \leq n} a_i e_i.$$

Then there holds

$$Ric_S(v) = \sum a_i a_j (Ric_S)_{ij}.$$

Applying the Gauss equation and the minimality of S , we deduce

$$\begin{aligned} (Ric_S)_{ij} &= \sum_{1 \leq k \leq n} R_{ikjk} + \sum_{1 \leq k \leq n} (A_{ij} A_{kk} - A_{ik} A_{jk}) \\ &= R_{ij} - R_{in+1jn+1} - \delta_{ij} A_{ii} A_{jj}, \end{aligned}$$

where R_{ijkl} is the Riemann curvature tensor of M . It follows that

$$Ric_S(v) = Ric(v) - K(v, \nu) - \sum_{1 \leq i \leq n} a_i^2 A_{ii}^2, \quad (3.2)$$

where for linearly independent tangent vectors v_1, v_2 at the same point of M , $K(v_1, v_2)$ denotes the sectional curvature of M determined by the plane spanned by v_1 and v_2 . On the other hand, by the choice of f , we have

$$-f^{-1} \Delta_S f = Ric(\nu) + |A|^2 + \lambda.$$

Hence we arrive at

Lemma 3 *The conformal Ricci curvature of $(\Omega, g|_S)$ in the direction of v is given by*

$$Ric_S^{(f,\sigma)}(v, v) = Ric(v) + \sigma Ric(\nu) - K(v, \nu) + \sigma |A|^2 - \sum_{1 \leq i \leq n} a_i^2 A_{ii}^2 + \sigma \lambda.$$

Lemma 4 *The following inequality holds*

$$\sum A_{ii}^2 \geq \frac{n}{n-1} \sum a_i^2 A_{ii}^2.$$

Consequently, we have

$$Ric_S^{(f,\sigma)}(v, v) \geq Ric(v) + \sigma Ric(\nu) - K(v, \nu) + \sigma \lambda,$$

provided that $\sigma \geq (n-1)/n$.

Proof By the minimality of S , we have

$$A_{11} + \dots + A_{nn} = 0.$$

Applying Cauchy-Schwarz inequality we then derive

$$A_{ii}^2 = \left(\sum_{j \neq i} A_{jj} \right)^2 \leq (n-1) \sum_{j \neq i} A_{jj}^2$$

for each i . Consequently,

$$a_i^2 A_{ii}^2 \leq (n-1) \sum_{j \neq i} a_i^2 A_{jj}^2.$$

Summing over i then yields

$$\sum a_i^2 A_{ii}^2 \leq (n-1) \sum_i A_{ii}^2 \left(\sum_{j \neq i} a_j^2 \right).$$

Since $\sum a_i^2 = 1$, adding $(n-1) \sum a_i^2 A_{ii}^2$ to both sides leads to the desired inequality. \square

Definition 2 Let σ be a positive number. For ordered orthogonal unit tangent vectors v_1, v_2 of M at the same point, we define the σ -weighted bi-Ricci curvature of M in the direction of v_1, v_2 to be

$$B_\sigma Rc(v_1, v_2) = Ric(v_1) + \sigma Ric(v_2) - K(v_1, v_2).$$

If $\sigma = \frac{n-1}{n} = \frac{m-2}{m-1}$, then $B_\sigma Rc$ is called the *harmonic bi-Ricci curvature* and denoted by $B_H Rc$.

Definition 3 We set $\partial S = \{ \lim p_k : p_k \in S, \{p_k\} \text{ does not converge in } S, \text{ but converges in } M \}$. If $\partial S = \emptyset$, we define $diam(S, \partial S)$ to be $diam S$. Otherwise we define it to be $\sup \{ dist(p, \partial S) : p \in S \}$

The following two theorems easily follow from Lemma 2, Lemma 2' and Lemma 4. (If $\partial S = \emptyset$, then they follow from Theorem 1, Theorem 1' and Lemma 4.)

Theorem 2 *Let σ satisfy $(n-1)/n \leq \sigma < 4/(n-1)$ ($(n-1)/n \leq \sigma \leq 4/(n-1)$ if $n = 3$, i.e. $m = 4$). Assume that M satisfies the curvature condition $B_\sigma Rc \geq \kappa$ for a constant κ . If \bar{S} is noncompact, then we require M to be complete. Moreover, we assume that S is connected and $\lambda(S) > -\frac{n}{n-1}\kappa$, where*

$$\lambda(S) = \inf \{ \lambda_\Omega : \Omega \text{ is a bounded domain in } S \text{ with } \bar{\Omega} \subset \overset{\circ}{S} \}.$$

Then

$$\text{diam}(S, \partial S), \text{dist}(Q, \partial \setminus Q) \leq c(n, \sigma) \frac{\pi}{\sqrt{\sigma \lambda(S) + \kappa}}, \quad (3.3)$$

where Q is an arbitrary subset of ∂S (if Q or $\partial S \setminus Q$ is empty, then we set $\text{dist}(Q, \partial S \setminus Q) = 0$), and

$$c(n, \sigma) = \sqrt{n-1 + \frac{(n-3)^2}{\frac{4}{\sigma} - n + 1}}$$

if $\sigma < \frac{4}{n-1}$, while $c(n, \sigma) = \sqrt{n-1}$ if $\sigma \leq \frac{4}{n-1}$ and $n = 3$.

Consequently, if κ is positive, S is stable, connected and 2-sided (i.e. its normal bundle is orientable), then

$$\text{diam}(S, \partial S), \text{dist}(Q, \partial S \setminus Q) \leq c(n, \sigma) \frac{\pi}{\sqrt{\kappa}}. \quad (3.4)$$

Remark 2 Note that the case of 1-weighted bi-Ricci curvature in dimensions $m \leq 5$ and the case of harmonic bi-Ricci curvature in dimensions $m \leq 6$ are covered by Theorem 2. This remark holds for all the results below.

Theorem 3 *Assume that $(n-1)/n \leq \sigma < 4/(n-1)$ ($(n-1)/n \leq \sigma \leq 4/(n-1)$ if $m = 4$), and there holds*

$$B_\sigma Rc \geq \frac{\kappa}{1 + \text{dist}(\cdot, p_0)^\delta}$$

for positive constants κ, δ with $\delta < 2$ and a point $p_0 \in M$. Let S be a connected, 2-sided stable minimal hypersurface in M . If \bar{S} is noncompact, then we require M to be complete. Then

$$\text{diam}(S, \partial S) \leq \sup \{ \theta_p^{-1} \left(\frac{c(m-1)\pi^2}{\kappa} \right) : p \in \partial S \}, \quad (3.5)$$

$$\text{dist}(B, \partial S \setminus B) \leq \sup\{\theta_p^{-1}(\frac{c(m-1)\pi^2}{\kappa}) : p \in B\}, \quad (3.6)$$

where $B \subset \partial S$.

We have the following corollaries.

Theorem 4 *Assume that M satisfy the curvature condition in Theorem 3. Let S be a complete, 2-sided stable minimal hypersurface in M . Then S is compact. Moreover, its universal cover is also compact, and hence it has finite fundamental group.*

Theorem 5 *Assume that M is an orientable, complete Riemannian manifold satisfying the curvature condition in Theorem 3. Let Γ be a smoothly embedded submanifold of codimension 2 in M . If it is homologically zero and orientable, then it bounds a compact smooth area-minimizing hypersurface.*

Proof By known results in geometric measure theory, there is an smoothly embedded, orientable area-minimizing hypersurface with boundary Γ . (It is obtained by minimizing mass for integral currents with Γ as boundary. Since integral currents are oriented, the resulting area-minimizing hypersurface is oriented.) Since M is orientable, it follows that S is 2-sided. Theorem 3 then implies that it is compact. \square

4 Topological implications

In this section we study topological implications of positive bi-Ricci curvature. First, for the convenience of the reader, we recall the following definition from [ShY1]:

Definition 4 Let M be a Riemannian manifold and \mathbf{V} a closed integral current (i.e. integer multiplicity rectifiable current) of codimension 2 in M (e.g. a finite union of closed, oriented submanifolds of codimension 2). (We only require \mathbf{V} to have locally finite mass.) Assume that \mathbf{V} is homologous to zero (in the class of integral currents of locally finite mass.) Then the *homology radius* of \mathbf{V} is defined to be

$$r_H(\mathbf{V}) = \sup \{r > 0, \mathbf{V} \text{ is not homologous to zero in its } r\text{-neighborhood}\}.$$

The homology radius $r_H(M)$ of M is defined to be the superior of $r_H(\mathbf{V})$ over all \mathbf{V} .

We shall denote the support of an integral current \mathbf{V} by $\text{supp}\mathbf{V}$.

Theorem 6 *Let M be an orientable, complete manifold of dimension m such that $B_\sigma R c \geq \kappa$ for a positive constant κ and a weight σ with $(m-2)/(m-1) \leq \sigma <$*

$4/(m-2)$ ($(m-2)/(m-1) \leq \sigma \leq 4/(m-2)$ if $m=4$). Then

$$r_H(M) \leq c(m-1, \sigma) \frac{\pi}{\sqrt{\kappa}}.$$

More generally, let M be an orientable, metrically complete manifold of dimension m , possibly with boundary, such that the above curvature condition holds. Let K be a subset of M containing ∂M and \mathbf{V} an integral current of codimension 2 in M , such that $[\mathbf{V}] = 0$ in $H_{m-2}(M, K)$ and $\text{dist}(\text{supp } \mathbf{V}, K) > c(m-1, \sigma) \frac{\pi}{\sqrt{\kappa}}$. Then \mathbf{V} must bound in the $c(m-1, \sigma) \frac{\pi}{\sqrt{\kappa}}$ -neighborhood of $\text{supp } \mathbf{V}$.

Proof As in the proof of Theorem 5 (and in [Sh1Y]), we find a mass-minimizing integral current \mathbf{W} with \mathbf{V} as boundary. The interior S of $\text{supp } \mathbf{W}$ is then a smoothly embedded, orientable stable minimal hypersurface with $\partial S = \text{supp } \mathbf{V}$. Applying Theorem 2 we then arrive at the desired conclusions. \square

Definition 5 Let M be a noncompact manifold of dimension m . For $1 \leq k \leq m-1$, let $H_k(M)_c$ denote the subgroup of $H_k(M)$ generated by chains with compact support. A class $\xi \in H_k(M)_c$ is called a *free class*, provided that it can be reduced to $H_k(M \setminus \Omega_j)_c$ for a sequence of increasing domains $\Omega_j \subset M$ with compact closure such that $M = \cup_j \Omega_j$.

Theorem 7 Let M be an orientable, complete Riemannian manifold of dimension m satisfying the curvature condition in Theorem 3. Then there is no nontrivial free class in $H_{m-2}(M)_c$. Consequently, if M is an orientable manifold of dimension m with $M = \bigcup_j \Omega_j$, such that there is a nontrivial free class in $H_{m-2}(M)_c$, then M carries no complete metric satisfying the curvature condition in Theorem 3.

Corollary 2 If M is diffeomorphic to the interior of a compact manifold \bar{M} with boundary such that the image of the inclusion $H_{m-2}(\partial M) \rightarrow H_{m-2}(\bar{M})$ is nonzero, then M carries no complete metric satisfying the curvature condition in Theorem 3.

Corollary 3 If $M = M' \times S^1$ or $M = M' \times \mathbf{R}^1$ such that M' is an orientable \mathbf{Z} -periodic manifold, then M carries no complete metric satisfying the curvature condition of Theorem 3.

Proof of Theorem 7 Assume the contrary. Then we can find integral currents with compact support \mathbf{V} and $\mathbf{V}_k, k = 1, 2, \dots$, such that neither of them is homologous to zero, each \mathbf{V}_k is homologous to \mathbf{V} , and $\text{dist}(\text{supp } \mathbf{V}, \text{supp } \mathbf{V}_k) \rightarrow \infty$. For each k , we can find a mass-minimizing integral current \mathbf{W}_k with boundary $\mathbf{V} - \mathbf{V}_k$. One

connected component S_k of the interior of $\text{supp}\mathbf{W}_k$ is a smoothly embedded, one-sided stable minimal hypersurfaces such that $\partial S_k = Q_k \cup T_k$ with $Q_k \neq \emptyset, T_k \neq \emptyset$ and $Q_k \subset \text{supp}\mathbf{V}, T_k \subset \text{supp}\mathbf{V}_k$. Applying (3.6) in Theorem 3 we then arrive at a contradiction.

□

Definition 6 We say that $M \in \mathcal{B}$, provided that M is a compact orientable manifold of dimension m such that

- (1) if \tilde{M} is a finite covering of M , then every nontrivial class in $H_{m-1}(M)$ can be represented by a finite disjoint union of embedded, compact orientable hypersurfaces (with multiplicities) which have finite fundamental groups,
- (2) if \tilde{M} is a noncompact covering of M , then there is no nontrivial free class in $H_{m-2}(\tilde{M})_c$.

Theorem 8 *Let M be a compact, orientable m -dimensional manifold of positive harmonic bi-Ricci curvature with $m \leq 6$ (or positive 1-weighted bi-Ricci curvature with $m \leq 5$). Then M belongs to the class \mathcal{B} .*

Since positive bi-Ricci curvatures imply positive scalar curvature, we note the following additional topological implications of positive bi-Ricci curvatures for a compact manifold M of dimension m :

- (1) if $m \leq 7$, then $M \in \mathcal{C}_m$, where \mathcal{C}_m is the class of manifolds introduced in [ScY2],
- (2) if M is spin, then its \hat{A} -genus and Hitchin invariant vanish,
- (3) if $m = 4$ and $b_2^+(M) > 1$ for one orientation of M , then the Seiberg-Witten invariants of M (for that orientation) vanish (if $b_2^+(M) = 1$, then the Seiberg-Witten invariants of M corresponding to the chamber of the given metric on M vanish).

The above results, along with the constructions of manifolds of positive bi-Ricci curvature in the next section, provide a basis for topological classification of manifolds of positive bi-Ricci curvatures in dimensions $m \leq 6$.

5 Constructions

We first make a few simple observations about the concept of bi-Ricci curvatures.

Observation 1 Summing over an orthonormal base we see that positive (σ -weighted) bi-Ricci curvature implies positive scalar curvature. On the other hand, positive sectional curvature obviously implies positive bi-Ricci curvature.

Observation 2 If M admits a metric with positive bi-Ricci curvature and positive

Ricci curvature (in particular, if M admits a metric with positive sectional curvature), then $M \times S^1$ and $M \times \mathbf{R}^1$ admits a metric with positive bi-Ricci curvature. Taking known examples of manifolds with positive sectional curvature, we then obtain manifolds which admit metrics with positive σ -weighted bi-Ricci curvature for all σ , but admit no metric with positive Ricci curvature.

Next we present our result about connected sums.

Theorem 9 *Let M_1, M_2 be two manifolds of the same dimension $m \geq 3$ such that they both admit metrics of positive σ -weighted bi-Ricci curvature for the same weight $\sigma > 0$. Then their connected sum $M_1 \# M_2$ also admits metrics of positive σ -weighted bi-Ricci curvature. Moreover, if M_1, M_2 both admit metrics whose σ -weighted bi-Ricci curvature is positive and satisfies a large scale condition (such as a growth condition), then $M_1 \# M_2$ also admits such metrics.*

This theorem clearly follows from Lemma 6 below.

Lemma 5 *Let (M, g) be a Riemannian manifold of dimension $m \geq 3$. Let $p \in M$ and B be a compact geodesic ball centered at p on which the σ -weighted bi-Ricci curvature for a weight $\sigma > 0$ is positive. Then there is a positive number ρ_0 with the following property. For arbitrary positive numbers $\rho \leq \rho_0$ and δ , there is a positive smooth function η on $M \setminus \{p\}$ such that $\eta|_{M \setminus B} \equiv 1$ and the conformal metric $\eta^2 g$ has positive σ -weighted bi-Ricci curvature on B . Moreover, measured in the C^2 norm and near $r = 0$ (r is the distance to p), the conformal metric $\eta^2 g$ is uniformly within δ distance from the standard product metric on $\mathbf{R}^+ \times S^{m-1}(\rho)$. Here $S^{m-1}(\rho)$ denotes the round $(m-1)$ -sphere of radius $\rho > 0$.*

Lemma 6 *Let (M, g) , σ , κ , p , B and ρ_0 be as above. For arbitrary positive number $\rho \leq \rho_0$, there is a smooth metric on $M \setminus \{p\}$, such that its σ -weighted bi-Ricci curvature is positive on $B \setminus \{p\}$, and near $r = 0$ it coincides with the standard product metric on $\mathbf{R}^+ \times S^{m-1}(\rho)$.*

Proof of Lemma 5 We follow a scheme in [MiW], where a connected sum theorem about positive isotropy curvature is proved. Set $\tau = \ln \eta$. We would like to determine τ so that the conformal metric $\tilde{g} = \eta^2 g$ has the desired properties. We choose $\tau = \tau(r)$. (We will make sure that τ is smooth.) Then

$$d\tau = \tau' dr, \nabla d\tau = \tau'' dr \otimes dr + \tau' \nabla dr.$$

Consider a unit tangent vector $v = ae_1 + b \frac{\partial}{\partial r}$, where e_1 is a unit tangent vector perpendicular to $\frac{\partial}{\partial r}$. By (2.2) and the above formulas, we have

$$\tilde{Ric}(v) = Ric(v) - \tau''(1 + (m-2)b^2) - \tau' \Delta r - (m-2)a^2 \tau' \nabla r(e_1, e_1)$$

$$-(m-2)a^2(\tau')^2.$$

But $\nabla dr(e_1, e_1) = 1/r + O(r)$, $\Delta r = (m-1)/r + O(r)$ for r small. Hence we have (for r small)

$$\begin{aligned}\tilde{Ric}(v) &= Ric(v) - \tau''(1 + (m-2)b^2) \\ &\quad - \tau'((m-1) + (m-2)a^2)\left(\frac{1}{r} + O(r)\right) - (m-2)a^2(\tau')^2.\end{aligned}\tag{5.1}$$

To proceed, we choose τ such that

$$\tau' = -\frac{\phi}{r}\tag{5.2}$$

for a function ϕ which is to be determined. Then

$$\tau'' = -\frac{\phi'}{r} + \frac{\phi}{r^2}.$$

It follows that

$$\begin{aligned}\tilde{Ric}(v) &= Ric(v) + \frac{\phi}{r^2}2(m-2)(a^2 + O(r^2)) \\ &\quad - (m-2)a^2\frac{\phi^2}{r^2} + \frac{\phi'}{r}(1 + (m-2)b^2).\end{aligned}\tag{5.3}$$

If $w = a_1e_2 + b_1\frac{\partial}{\partial r}$ is another unit tangent vector such that $e_2 \perp \frac{\partial}{\partial r}$ and $v \perp w$, then we have

$$\begin{aligned}\tilde{Ric}(v) + \sigma\tilde{Ric}(w) &= Ric(v) + \sigma Ric(w) + \frac{\phi}{r^2}2(m-2)(a^2 + \sigma a_1^2 + O(r^2)) \\ &\quad - (m-2)(a^2 + \sigma a_1^2)\frac{\phi^2}{r^2} + \frac{\phi'}{r}(1 + (m-2)(b^2 + \sigma b_1^2)).\end{aligned}\tag{5.4}$$

On the other hand, the following law holds for Riemann curvature tensor (see [B]),

$$e^{-2\tau}\tilde{R}m = Rm - g\Diamond(\nabla d\tau - d\tau \otimes d\tau + \frac{1}{2}|\nabla\tau|^2g),\tag{5.5}$$

where for symmetric 2-tensors A, B ,

$$\begin{aligned}(A\Diamond B)(v_1, w_1, v_2, w_2) &= A(v_1, v_2)B(w_1, w_2) + A(w_1, w_2)B(v_1, v_2) \\ &\quad - A(v_1, w_2)B(w_1, v_2) - A(w_1, v_2)B(v_1, w_2).\end{aligned}$$

Simple computations then yield for $\tilde{v} = e^{-\tau}v$, $\tilde{w} = e^{-\tau}w$

$$\begin{aligned}\tilde{R}m(v, w, \tilde{v}, \tilde{w}) &= K(v, w) - \nabla d\tau(v, v) - \nabla d\tau(w, w) \\ &\quad + |v\tau|^2 + |w\tau|^2 - |\nabla\tau|^2.\end{aligned}\tag{5.6}$$

Consequently,

$$\begin{aligned}\tilde{R}m(v, w, \tilde{v}, \tilde{w}) &= K(v, w) + \frac{\phi}{r^2} 2(a^2 + a_1^2 - 1)(1 + O(r^2)) \\ &\quad - \frac{\phi^2}{r^2} (a^2 + a_1^2) + \frac{\phi'}{r} (b^2 + b_1^2).\end{aligned}\tag{5.7}$$

Combining this with (5.4) we then deduce

$$\begin{aligned}e^{2\tau} B_\sigma \tilde{R}c(\tilde{v}, \tilde{w}) &= B_\sigma Rc(v, w) + \frac{\phi}{r^2} 2(1 - a_1^2 + (m - 3)a^2 + (m - 2)\sigma a_1^2 + O(r^2)) \\ &\quad - \frac{\phi^2}{r^2} ((m - 2)(a^2 + \sigma a_1^2) - a^2 - a_1^2) + \frac{\phi'}{r} (1 - b_1^2 + (m - 3)b^2 + (m - 2)\sigma b_1^2).\end{aligned}$$

Since $1 - a_1^2 + (m - 3)a^2 + (m - 2)\sigma a_1^2 \geq \min\{1, \sigma\}(1 - a_1^2 + (m - 2)a_1^2) \geq \min\{1, \sigma\}$, this formula implies that

$$e^{2\tau} B_\sigma \tilde{R}c(\tilde{v}, \tilde{w}) \geq B_\sigma Rc(v, w) + \frac{1}{r^2} (c_0 - c_1 \phi^2) r^2 + c_1 \frac{\phi'}{r},\tag{5.8}$$

provided that $r \leq r_0$ for a positive number r_0 and $\phi' \leq 0$, where $c_0 = \min\{1, \sigma\}/2$ and $c_1 = m + (m - 2)\sigma$. Let κ denote the minimum of $B_\sigma Rc$ on B . We choose r_0 such that $B_{r_0} \subset B$. Then we seek a nonincreasing ϕ such that

$$\kappa + \frac{1}{r^2} \phi (c_0 - c_1 \phi) + c_1 \frac{\phi'}{r} > 0.\tag{5.9}$$

Consider the change of variables $r = r_0 e^{-t}$ mapping the interval $[0, \infty)$ onto $(0, r_0]$. Then (5.9) is equivalent to

$$c_1 \psi' < \kappa r_0^2 e^{-2t} + \psi (c_0 - c_1 \psi)\tag{5.10}$$

with $\phi(s) = \psi(r_0 e^{-t})$ (we require $\psi' \geq 0$). Rescaling, we can reduce this to

$$\psi' < c_1^{-1} \kappa r_0^2 e^{-2t} + \psi (c_1^{-1} c_0 - \psi)\tag{5.11}$$

Observe that the solutions of the ODE $\psi' = \psi (c_1^{-1} c_0 - \psi)$ are given by

$$\psi_c(t) = \frac{c_1^{-1} c_0 c e^{c_1^{-1} c_0 t}}{1 + c e^{c_1^{-1} c_0 t}}$$

for arbitrary constants c . For each positive number $t_1 > 2 \ln 2$, choose a smooth non-decreasing function $\beta = \beta_{t_1}$ such that $\beta(t) \equiv 0$ for $t \leq \ln 2$, $\beta(t) = e^{c_1^{-1} c_0 t}$ for $2 \ln 2 \leq t \leq t_1$ and $\beta(t) \equiv e^{c_1^{-1} c_0 (t_1 + 1)}$ for $t \geq t_1 + 1$. We set

$$\tilde{\psi}_c(t) = \frac{c_1^{-1} c_0 c \beta(t)}{1 + c \beta(t)}.$$

Since κ is positive, it is easy to see that for each t_1 , if we choose $c = c(t_1)$ to be small enough, then $\tilde{\psi}$ will satisfy the inequality (5.11) (and hence (5.10) after scaling). The corresponding ϕ then satisfies (5.9), equals 1 near 0, and equals 0 for $r \geq r_0/2$. For this choice of ϕ , we have $B_\sigma \tilde{R}c > 0$ on B . Furthermore, by (5.2), there holds

$$e^{2\tau(r)} = e^{2\tau(r_1)} \frac{r_1^2}{r^2}, \quad (5.12)$$

$$\tau(r_1) = \int_{r_1}^{r_0} \frac{\phi}{r}, \quad (5.13)$$

where $r_1 = r_0 e^{-(t_1+1)}$. Since g is asymptotically euclidean at p , (5.12) implies that near $r = 0$ or $s = \infty$ under the transformation $s = -\ln r$, the conformal metric \tilde{g} approaches the metric $e^{2\tau(r_1)} \tilde{r}_1^2 (ds^2 + d\omega^2)$, where $d\omega^2$ denotes the metric of the unit sphere. But the latter is equivalent to the product metric on $\mathbf{R}^+ \times S^{m-1}(\rho)$, where $\rho = e^{\tau(r_1)} r_1$. One easily works out the explicit formula for ρ and shows that it can be made arbitrarily small by choosing t_1 large. □

Proof of Lemma 6 We interpolate the product metric on $\mathbf{R}^+ \times S^{m-1}(\rho)$ with \tilde{g} . □

6 Minimal surfaces of higher codimensions

First, we present an extension of the concept of conformal Ricci curvature and the generalized Bonnet-Myers theorem. Consider a Riemannian manifold (N, h) of dimension n .

Definition 7 Let k be a natural number, f_1, \dots, f_k positive smooth functions on N and a_1, \dots, a_k positive numbers. The conformal Ricci tensor of (N, h) associated with the conformal factor $\mathbf{f} = (f_1, \dots, f_k)$ and weight $\mathbf{a} = (a_1, \dots, a_k)$ is defined to be

$$Ric(\mathbf{f}, \mathbf{a}) = Ric - \left(\sum f_i^{-1} \Delta f_i \right) h.$$

For a conformal factor \mathbf{f} and a weight \mathbf{a} , we consider the conformal metric $\tilde{h} = f_1^{2a_1} \dots f_k^{2a_k} h$. Using (2.5) with f^σ replaced by $\eta = f_1^{a_1} \dots f_k^{a_k}$ we deduce

$$Ric(\mathbf{f}, \mathbf{a})(v, v) = \tilde{Ric}(v, v) + (n-2) \frac{\partial^2 \ln \eta}{\partial s^2} - \sum a_i |\nabla \ln f_i|^2 \quad (6.1)$$

for unit tangent vectors v , where we use a curve $\gamma(s)$ in a similar way to the context of (2.5). Next let $\gamma : [0, l] \rightarrow N$ be a unit speed (measured in h) curve which minimizes length in \tilde{h} up to second order. Arguing as in the proof of Lemma 1 we infer

$$\begin{aligned} (n-1) \int \left(\frac{d\phi}{ds}\right)^2 + \frac{n-1}{4} \phi^2 \left(\sum \frac{d \ln f_i^{a_i}}{ds}\right)^2 + (3-n) \int \phi \frac{d\phi}{ds} \frac{d \ln \eta}{ds} \\ \geq \int \phi^2 Ric(\mathbf{f}, \mathbf{a}) \left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial s}\right) + \int \phi^2 \left(\sum \frac{1}{a_i} |\nabla \ln f_i|^2\right). \end{aligned} \quad (6.2)$$

Now, the arguments in the proof of Lemma 2 and Cauchy-Schwarz inequality lead to

Lemma 7 *Assume $a \equiv \sum a_i < \frac{4}{n-1}$ ($a \leq \frac{4}{n-1}$ if $n = 3$) and $Ric(\mathbf{f}, \mathbf{a}) \geq \kappa$ for a positive constant κ . Then we have*

$$l \leq \sqrt{n-1 + \frac{(n-3)^2}{\frac{4}{a} - n + 1} \frac{\pi}{\sqrt{\kappa}}} \quad (6.3)$$

if $a < \frac{4}{n-1}$, and

$$l \leq \sqrt{n-1} \frac{\pi}{\kappa} \quad (6.4)$$

if $a \leq \frac{4}{n-1}$ and $n = 3$.

Of course, all the statements in Lemma 2, Lemma 2', Theorem 2 and Theorem 2' can be extended to the present situation. We leave the details to the reader.

Next let (M, g) be a Riemannian manifold of dimension $m > n$ and $S \subset M$ an immersed stable minimal surface of dimension n . We have the following formula for the second variation of area [Si]

$$I(X) = \int_S \left(\sum |(\nabla_{e_i} X)^\perp|^2 - \sum \langle X, A(e_i, e_j) \rangle^2 - \sum Rm(e_i, X, e_i, X) \right), \quad (6.5)$$

where X is a normal vector field and $\{e_i\}$ a local orthonormal tangent base of S . Consider a bounded domain Ω such that $\bar{\Omega} \subset \overset{\circ}{S}$ and there is a smooth unit normal vector field ν on $\bar{\Omega}$. Choosing $X = \phi\nu$ in (6.1) we obtain

$$\int_S (|\nabla \phi|^2 + \phi^2 |\nabla^\perp \nu|^2 - \phi^2 |A_\nu|^2 - \sum Rm(e_i, \nu, e_i, \nu) \phi^2) \geq 0$$

for all smooth functions ϕ with compact support in Ω , where ∇^\perp denotes the normal connection and $A_\nu = A \cdot \nu$. Let f be a positive (in Ω) first eigenfunction of the operator $L\phi = -\Delta_S \phi + |\nabla^\perp \nu|^2 \phi - |A_\nu|^2 \phi - \sum Rm(e_i, \nu, e_i, \nu) \phi$. Then $Lf \geq 0$. Consider the Riemannian manifold $N = \Omega$ with metric $g|_S$. We compute the conformal Ricci

curvature $Ric^{(f,\sigma)}(v, v)$ for a weight σ and a unit tangent vector v . Choose local orthonormal normal frame ν_α . Then we have by the Gauss equation

$$Ric_S(v) = \sum_i K(v, e_i) + \sum_\alpha \sum_i (A_{\nu_\alpha}(v, v) \cdot A_{\nu_\alpha}(e_i, e_i) - A_{\nu_\alpha}(v, e_i) \cdot A_{\nu_\alpha}(v, e_i)).$$

For each fixed α , the quantity $\sum_i (A_{\nu_\alpha}(v, v) \cdot A_{\nu_\alpha}(e_i, e_i) - A_{\nu_\alpha}(v, e_i) \cdot A_{\nu_\alpha}(v, e_i)) = -\sum_i A_{\nu_\alpha}(v, e_i) \cdot A_{\nu_\alpha}(v, e_i)$ (by the minimality) is independent of the choice of e_i . We switch to a base $\{e_{i,\alpha}\}$ to diagonalize A_{ν_α} . In this base, we have $v = \sum_i a_{i,\alpha} e_{i,\alpha}$ for suitable $a_{i,\alpha}$. It follows that

$$Ric_S(v) = \sum_i K(v, e_i) - \sum_\alpha \sum_i a_{i,\alpha}^2 A_{\nu_\alpha}(e_{i,\alpha}, e_{i,\alpha})^2. \quad (6.6)$$

Applying Lemma 4 we arrive at

Lemma 8 *There holds*

$$\begin{aligned} Ric^{(f,\sigma)}(v, v) &\geq \sum K(v, e_i) \sigma \sum K(\nu, e_i) \\ &\quad + \sigma |A_\nu|^2 - \frac{n-1}{n} |A|^2 - \sigma |\nabla^\perp \nu|^2. \end{aligned} \quad (6.7)$$

Now we assume that there is a global smooth orthonormal frame $\{\nu_\alpha\}$ for the normal bundle over $\bar{\Omega}$. For each $\nu = \nu_\alpha$ we choose a positive first eigenfunction f_α . Then we obtain

Lemma 9 *For $\mathbf{f} = (f_1, \dots, f_{m-n})$ there holds*

$$\begin{aligned} Ric^{(\mathbf{f}, \mathbf{a})}(v, v) &\geq \sum_i K(v, e_i) + \sum_\alpha \sigma_\alpha (\sum_i K(\nu_\alpha, e_i)) \\ &\quad + \sum_\alpha \sigma_\alpha |A_{\nu_\alpha}|^2 - \frac{n-1}{n} |A|^2 - \sum_\alpha \sigma_\alpha |\nabla^\perp \nu_\alpha|^2. \end{aligned} \quad (6.8)$$

We choose $a_\alpha = \sigma$ for all α and a positive σ . We can apply Lemma 7 and Lemma 9 to the situation $(n-1)/n < \sigma < 4/((m-n)(n-1))$ (or $(n-1)/n < \sigma \leq 4/(m-n)(n-1)$ if $n=3$). There are two possible cases (besides the already treated case $m=n+1$): *Case 1* $n=2, 4 \leq m \leq 9$ and $1/2 < \sigma < 4/(m-n)$; *Case 2* $n=3, m=5$ and $2/3 < \sigma \leq 1$.

Definition 8 Let $p \in M$ and V be a nontrivial proper subspace of $T_p M$. For a unit tangent vector $v \in V$ and a positive weight σ we set

$$K_\sigma(v, V) = \sum_i K(v, e_i) + \sigma \sum_{i, \alpha} K(\nu_\alpha, e_i),$$

where $\{e_i\}$ denotes an orthonormal base of V and $\{\nu_\alpha\}$ an orthonormal base of V^\perp . (This expression is independent of the choice of the bases.)

We leave the version of the following result based on Lemma 2' (instead of Lemma 2) to the reader.

Proposition 1 *Assume that we are in one of the above two cases. Moreover, assume that there is a positive constant κ such that $K_\sigma(v, V) \geq \kappa$ for all V and $v \in V$ with $\dim V = n$. Let S' be a connected domain of S such that the normal bundle of S' is flat and trivial. More generally, assume that the normal bundle of S' admits a global orthonormal frame $\{\nu_\alpha\}$ which is “almost flat” in the sense that*

$$\sum |\nabla^\perp \nu_\alpha(p)|^2 \leq (\sigma - \frac{n-1}{n})|A|^2 + \epsilon(1 + \text{dist}^2(p, \partial S')) \quad (6.9)$$

for a positive constant ϵ with

$$\epsilon < \frac{\kappa^2}{2(\kappa + 1 + \frac{(m-2)\sigma}{4-(m-2)\sigma}\pi^2)} \quad (6.10)$$

in Case 1, or

$$\epsilon < \frac{\kappa}{8\pi + 2\kappa} \quad (6.11)$$

in Case 2.

Then we have for $Q \subset \partial S'$

$$\text{diam}(S', \partial S'), \text{dist}(Q, \partial S' \setminus Q) \leq \sqrt{1 + \frac{(m-2)\sigma}{4-(m-2)\sigma} \frac{\sqrt{2\pi}}{\sqrt{\kappa-2\epsilon}}} \quad (6.12)$$

in Case 1, or

$$\text{diam}(S', \partial S'), \text{dist}(Q, \partial S' \setminus Q) \leq \frac{2\pi}{\sqrt{\kappa-2\epsilon}} \quad (6.13)$$

in Case 2.

Proof We prove the estimate for $\text{diam}(S', \partial S')$ in Case 2. The other estimate and case can be handled in a similar way. First assume that S' is bounded and $\bar{S}' \subset$

\mathring{S} . Consider the positive first eigenfunctions f_α on S' corresponding to ν_α and the associated conformal metric $\tilde{g} = f_1^{2\sigma} \cdots f_{m-n}^{2\sigma} g_S$. For small $t > 0$, consider an arbitrary connected subdomain Ω in S' such that $\bar{\Omega}$ is contained in the interior of S' and $\text{dist}(p, \partial S') \leq t$ for all $p \in \partial\Omega$.

Set $F = \{p \in \bar{\Omega} : \text{dist}(p, \partial\Omega) \leq \sqrt{\frac{\kappa}{2\epsilon}} - t\}$. We claim that F is open in $\bar{\Omega}$. It suffices to show that no point in $\bar{\Omega}$ has distance $\sqrt{\frac{\kappa}{2\epsilon}} - t$ to $\partial\Omega$. Assume the contrary and let $p \in \bar{\Omega}$ has this distance to $\partial\Omega$. Let $\tilde{\gamma}$ be a minimizing geodesic from p to $\partial\Omega$ with respect to the metric \tilde{g} and $\gamma : [0, l] \rightarrow \bar{\Omega}'$ its reparametrization with unit tangent vector measured in g_S . By Lemma 7 and the almost flatness assumption, we deduce $(\mathbf{a} = (\sigma, \dots, \sigma))$

$$\text{Ric}(\mathbf{f}, \mathbf{a})\left(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial s}\right) \geq \frac{\kappa}{2} - t\epsilon.$$

Applying Lemma 7 we then infer

$$l \leq \frac{2\pi}{\sqrt{\kappa - 2\epsilon}} < \sqrt{\frac{\kappa}{2\epsilon}} - t\epsilon,$$

provided that t is small enough. This is a contradiction. Since F is also closed, it follows that $F = \Omega$. Approximating S' by Ω we then obtain the desired estimate. If S' is not bounded, we can approximate it by bounded domains. \square

Remark 3 We can replace the term $\text{dist}^2(p, \partial\Omega)$ in (6.9) by other increasing functions of $\text{dist}(p, \partial\Omega)$. Of course, the requirements (6.10) and (6.11) on ϵ will change accordingly.

Remark 4 The triviality condition on the normal bundle in Proposition 1 is not so restrictive as it might appear at the first glance. Indeed, for example, if S is topologically a sphere, then $S' = S \setminus \{p\}$ for any $p \in S$ satisfies this condition. In contrast, the almost flatness condition is more serious. We would like to point out that if the normal curvature is small, then it is easy to construct almost flat frames in an injectivity domain of S . On the other hand, by the Ricci equation, small normal curvature follows from a suitable pinching condition on the curvature tensor of M . As a consequence, some local size estimates for S can be derived under such a pinching condition. For example, we can deduce an upper bound for the conjugate radius at every point. By the Gauss equation, this implies an estimate for the second fundamental form in the case $\dim S = 2$. Details will be given elsewhere.

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